COMPSCI 230: Discrete Mathematics for Computer Science	3/05/2020

Recitation 9: Partial and Total Orders

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1 Recall definitions

- **Injective:** R is *injective* if and only if all elements of B have in degree at most 1 (≤ 1) - each element on the domain side of the bipartite graph has in-degree at most 1. "vertical line test"
- **Surjective:** R is *surjective* if and only if all elements of B at in degree at least 1 - each element on the domain side of the bipartite graph has in-degree at least 1. "horizontal line test"
- **Bijective:** R is *bijective* if and only if all elements of B have in degree exactly 1 - each element on the domain side of the bipartite graph has in-degree exactly 1. R is *bijective* if and only if R is *injective* and *surjective*.
- **Reflexivity** R is *reflexive* if and only if $\forall x \in A.xRx$. In terms of the graph, every vertex has a self-loop.
- **Irreflexivity** R is *irreflexive* if and only if $\forall x \in A.\neg(xRx)$. In terms of the graph, no vertex has a self-loop.
- **Symmetry** R is *symmetric* if and only if $\forall x, y \in A.xRy \rightarrow yRx$. In terms of the graph, every edge from x to y has an edge back from y to x.
- **Asymmetry** R is *asymmetric* if and only if $\forall x, y \in A.xRy \rightarrow \neg(yRx)$.
- **Antisymmetry** R is *antisymmetric* if and only if $\forall x, y \in A.x \neq y \wedge xRy \rightarrow \neg(yRx)$. Same as asymmetry, but with self loops.
- **Transitivity** R is *transitive* if and only if $\forall x, y, z \in A.xRy \wedge yRz \rightarrow xRz$.
- **Partial Order** R is a *partial order* if and only if it is transitive and asymmetric
  - **Weak Partial Order** R is a *weak partial order* if and only if it is transitive and antisymmetric (we are not learning this definition)
  - **Strong Partial Order** R is a *strong partial order* if and only if it is transitive and asymmetric (this is the definition of partial order we are using in this class)
- **Total Order** R is a *total order* if and only if it is a partial order, and $\forall a, b \in A,\ (a, b) \in R$ or $(b, a) \in R$.
- **Equivalence Relation** R is an *equivalence relation* if and only if it is reflexive, transitive, and symmetric.
- **Equivalence Class** The *equivalence class* of an element $a \in A$ is denoted $[a]_R$ and is the set of all elements that relate to a under R. In other words, the equivalence class of a is the image $R(a)$.
- **Chain** A *chain* is a subset of elements in A such that any two elements in the subset are related to each other.

2 Warm-up (review): Provide functions $f : \mathbb{Z} \to \mathbb{Z}^+$ with the following properties.

1. f is neither surjective, nor injective
2. f is surjective and not injective
3. f is injective and not surjective
4. f is both injective and surjective

Solution

1. f is neither surjective, nor injective : $f(x) = 1$
2. f is surjective and not injective : $f : \mathbb{R} \to \mathbb{Z}^+ \setminus \{0\}$ $f(x) = |x| + 1$
3. f is injective and not surjective :
   $f(x) = \begin{cases} 2x + 3 \text{ if } x \geq 0 \\ -2x \text{ else} \end{cases}$

   Proof. First, we prove that f is not surjective. This consists of showing that the in-degree of some $y \in \mathbb{Z}^+$ is zero. In particular, this is true for (only) $y = 1$. To see this, suppose there was such an $x \in \mathbb{Z}$ such that $f(x) = 1$ for sake of contradiction. If $x \geq 0$, then $f(x) = 2x + 3 \geq 3$, which is a contradiction. Otherwise, if $x < 0$, then $x \leq -1$ and thus $f(x) \geq 2$ which is a contradiction. Thus, the in-degree of 1 is zero. To prove that f is injective, we show that the in-degree of every $y \in \mathbb{Z}^+$ is at most one. We've shown above that the in-degree of 1 is zero, so we only have to show this for the rest of the positive integers – those greater than two. First see that $f(x)$ is odd for any non-negative integer x, and that f (x) is even for any negative integer x. Now suppose, for sake of contradiction, that there is some positive integer y that is greater than two which has in-degree at least two. Then there exists distinct $a, b \in \mathbb{Z}$ such that $f(a) = f(b) = y$. From the previous observation, if y is odd, then $a, b \geq 0$, and otherwise if y is even, then $a, b < 0$. In the former case, then $f(a) = f(b)$ implies $2a + 3 = 2b + 3$ which only holds when $a = b$, a contradiction. In the latter case, then $f(a) = f(b)$ implies $-2a = -2b$ which is only holds when $a = b$, a contradiction. Thus, the in-degree of all $y \in \mathbb{Z}$ is at most one, so f is surjective □

4. f is both injective and surjective:
   $f(x) = \begin{cases} 2x + 1 \text{ if } x \geq 0 \\ -2x \text{ else} \end{cases}$

   Proof. A similar proof for the injectivity of the function in part c implies that f is injective. The details are omitted here. To see that f is surjective, we show that the in-degree of every $y \in \mathbb{Z}^+$ is at least one. Consider an arbitrary $y \in \mathbb{Z}^+$. If y is odd, see that $y - 1$ is even and non-negative, so $(y - 1)/2$ is non-negative and integral. Thus, $f((y - 1)/2) = y$. If y is even, then $-y/2$ is negative and integral. Thus, $f(-y/2) = y$. In either case, there is some $x \in \mathbb{Z}$ such that $f(x) = y$, so the in-degree of all $y \in \mathbb{Z}^+$ is at least one. We conclude that f is surjective. □

3 For each of the following, state whether it is a strong partial order or not. If not, state which axiom it violates.

1. The order of getting dressed in the morning. Namely, $A =$ set of clothes, and $aRb \leftrightarrow$ we must put on a before we put on b. This example is a helpful, easy, toy example.
2. The superset relation $\supseteq$ on the power set pow$(\{1,2,3,4\})$
3. The superset relation $\supset$ on the power set pow$(\{1,2,3,4\})$
4. The relation between two non negative integers given by $a \equiv b \mod 8$
5. The relation between two propositional formulas A and B in $A \to B$ (implies).
6. The relation ”beats” in rock paper scissors
7. The empty relation on the set of integers
8. The identity relation on the set of integers

Solution

1. The order of getting dressed in the morning. Namely, $A =$ set of clothes, and $aRb \leftrightarrow$ we must put on a before we put on b. This example is a helpful, easy, toy example.
   Solution Strong partial order.
2. The superset relation $\supseteq$ on the power set pow$(\{1,2,3,4\})$
   Solution For our purposes, this is not a partial order as we define partial orders as asymmetric, not antisymmetric. In fact, is a weak partial order - every element is related to itself and it is transitive. Formally showing antisymmetry is as follows. Let $A \supseteq B$ and $A \neq B$. Then $\exists x \in A.x \notin B$. Then $\neg(B \supseteq A)$.
3. The strict superset relation $\supset$ on the power set pow$(\{1,2,3,4\})$
   Solution This is a strong partial order.
4. The relation between two non negative integers given by $a \equiv b\text{mod } 8$
   Solution Neither. The relation is symmetric.
5. The relation between two propositional formulas A and B in $A \to B$ (implies).
   Solution Weak partial order - every element is related to itself and it is transitive. For our purposes, however, this is not a partial order as we only learned the definition of a strong partial order.
6. The relation ”beats” in rock paper scissors
   Solution Neither - the relation is not transitive.
7. The empty relation on the set of integers
   Solution The relation is transitive. The relation is not reflexive. The relation is symmetric. BUT the relation is also vacuously asymmetric, so it is correct to say the empty relation is a strong partial order.

8. The identity relation on the set of integers : $Id(A) = \{(a, b) \in A \times A | a = b\}$
   Solution The identity set is symmetric, so is not a strong partial order.

4 Consider the relation R on the set $A = \{n \in \mathbb{Z} | 1 \leq n \leq 10\}$:

$$R = \{(x, y) \in A \times A | (x = y) \vee ((x \text{ is odd}) \wedge (x < y))\}$$

1. Is this a (strong) partial order?
   Solution No, this is a weak partial order - what we are learning as not a partial ordering. Can be seen since $R \supset \{(a, b) \in A \times A | x = y\}$, which is not a partial order by above.
2. Consider now $R' = \{(x, y) \in A \times A | ((x \text{ is odd}) \wedge (x < y))\}$
3. Is this a (strong) partial order?
   Solution Yes. The relation is a subset of $\{(x, y) \in A \times A | x < y\}$, which is a symmetric relation. Further, this is a total partial order on the set given. Therefore, R must be asymmetric. It is easy to verify transitivity.

5 Indicate which of the following are strong partial orders, or an equivalence relation. If neither, state which of the following properties it has : transitive, reflexive, symmetric and asymmetric

1. The relation on the integers $a = b + 1$
2. The ”relatively prime” relation on the non negative integers
3. The relation ”has the same prime factors” on the non negative integers.

6 Let $A = \mathbb{R}^3$. Let $R = \{(a, b) \in A \times A | a_3 = b_3\}$ (two elements relate to each other iff they have the same z value. Prove that R is an equivalence relation.

Solution

Proof. We must show reflexivity, transitivity and symmetry. Consider $a \in \mathbb{R}^3$. Clearly $(a, a) \in R$. Consider $a, b, c \in A | (a, b) \in R \wedge (b, c) \in R$. Then, $a_3 = b_3 = c_3$. Then, $a_3 = c_3$, so $(a, c) \in R$. Symmetry follows in a similar fashion. □

7 In an n-player round-robin tournament, every pair of distinct players compete in a single game. Assume that every game has a winner—there are no ties. The results of such a tournament can then be represented with a tournament digraph where the vertices correspond to players and there is an edge $x \to y$ iff "x beat y" in their game

1. Briefly explain why a tournament digraph cannot have cycles of length one or two.
2. Briefly explain whether the "beats" relation is always/sometimes/never symmetric, reflexive, irreflexive, transitive.
3. If a tournament graph has no cycles of length three, prove that it is a partial order.

Solution

1. Briefly explain why a tournament digraph cannot have cycles of length one or two.
   Solution x cannot beat themselves, so there are no cycles of length one. Either x beats y or vice versa; we cannot have both since every pair of distinct players only play one game against each other.
2. Briefly explain whether the "beats" relation is always/sometimes/never symmetric, reflexive, irreflexive, transitive.
   Solution The relation is never reflexive - no one can beat themselves. The relation is never symmetric, as there are no cycles of length two by above. If, in all cases where a beats b and b beats c, we have that a beats c, the relation is transitive.
3. If a tournament graph has no cycles of length three, prove that it is a partial order.
   Solution It suffices to show that if the relation is not a partial order, there must be a cycle of length three by contrapositive. Either R is not asymmetric, or R is not transitive. If R is not transitive, then $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$. But since a must have played c, this means $(c, a) \in R$. Thus, there would be a cycle of length three. On the other hand, we know that there are no pairs a, b such that $(a, b) \in R$ and $(b, a) \in R$ since a and b play each other only once, and so R must be asymmetric.

8 Prove that if a binary relation R is transitive and irreflexive, then it is asymmetric.

Solution

Proof. Suppose not, so that R is transitive and irreflexive and symmetric. Then, there exists $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$. Then, by transitivity $(a, a) \in R$, which contradicts irreflexivity. □

9 A set of functions $f, g \colon D \to \mathbb{R}$ can be partially ordered by the relation $<$ where $f < g \leftrightarrow \forall d \in D, f(d) < g(d)$. Describe a set of functions and infinite chain of functions in that set. (If having trouble, can give hint: consider linear functions of the form $f(x) = ax + b$, a, b constants.)

Solution The set of linear functions as above is the set needed, and we can define a chain by the set of linear functions with the same slope.

10 Let A be a set, and R a relation on that set. Prove or disprove the following.

1. There exists an equivalence relation S on A such that $R \subseteq S$.
2. There exists a (strong) partial order such that $S \subseteq R$.

Solution

1. **Solution** Let $S = A \times A$. Verification of the properties is left as an exercise.
2. **Solution** Let $S = \emptyset$. Verification of properties is left as an exercise.